Introduction to Elliptic PDEs

Part II.

for online PDE coffee chat.

Mark Ma

Problem ③ Continued

What have we done so far? for $0<\alpha<1$.

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

for $f \in C^{0,\alpha}$ compactly supported, $g \in C(\bar{\Omega})$, $\partial\Omega$ satisfying exterior ball condition. we've established $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ unique solution. Moreover, using Singular integral approach, we gain interior $C^{2,\alpha}$ estimates.

We have shown that for $f \in C_0^{0,\alpha}$ $Tf := p.v.(\partial_{ij}\Gamma * f) \in C^{0,\alpha}$.

thus using formula $\partial_{ij} w = \frac{\delta_{ij}}{n} f + p.v.(\partial_{ij}\Gamma * f) \Rightarrow w \in C^{2,\alpha}_{loc}$.

In particular writing $u = w + v$ for $w = \int_{\mathbb{R}^n} \Gamma(x-y) f(y) dy$ and v harmonic. one obtain

Interior Estimates on Balls

Let $u \in C^2(B_1)$ solve $\Delta u = f$ in B_1 with $f \in C_0^{0,\alpha}(\bar{B}_1)$

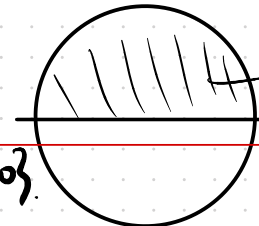
then one has estimate

$$[u]_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C(n,\alpha)\left(\|u\|_{C^0(\bar{B}_1)} + \|f\|_{C^{0,\alpha}(\bar{B}_1)} \right)$$

$\|u\|_{C^0}$ on RHS comes from bounding the harmonic function $v = u - w$.

Today we start with **Global Regularity** and ask whether our unique solution lies in $C^{2,\alpha}(\overline{\Omega})$?

Step 1 Boundary Estimates on half Balls



this upper half Ball we denote B_1^+.

Lemma let $u \in C^2(B_1^+) \cap C^0(\overline{B_1^+})$, $u=0$ on $\{x_n=0\}$.

Solve $\Delta u = f$ B_1^+. with $f \in C^{0,\alpha}(\overline{B_1^+})$ with compact support in B_1.

then we want estimate $[u]_{C^{2,\alpha}(\overline{B^+_{1/2}})} \le C(n,\alpha)\left(\|u\|_{C^0(\overline{B_1^+})} + \|f\|_{C^{0,\alpha}(\overline{B_1^+})} \right)$ (*)

All bad things happen because there could be "cut" in the force term.

our force f.

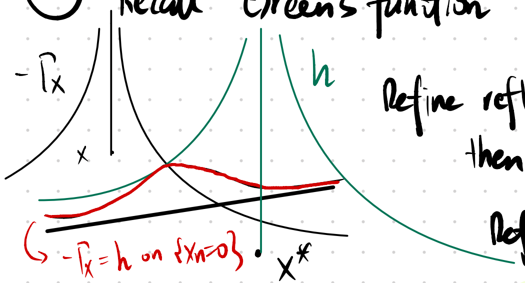
strategy: to extend force, Newtonian Potential and Harmonic Part.

Morally we still want $u = w + v$, $w = \Gamma * f$, v harmonic function.

Leverage $u = 0$ $\{x_n = 0\}$. design $\tilde{w}$ so $\tilde{w} = 0$ on $\{x_n = 0\}$. thus $\tilde{v} = u - \tilde{w} = 0$ on $\{x_n = 0\}$.

use Green's function — use Schwartz Reflection.

(1) Recall Green's function $G_{\mathbb{R}^n_+}(x,y) := \Gamma_x(y) + h(y)$ where $\begin{cases} \Delta h = 0 & \mathbb{R}^n_+ \\ h = -\Gamma_x & \{x_n = 0\}. \end{cases}$



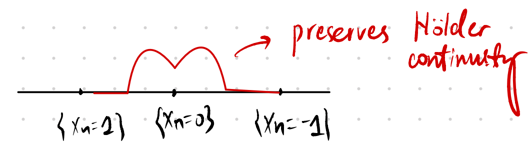
Define reflection point $x^* = (x_1, \cdots, x_{n-1}, -x_n)$ $\forall\, x \in \mathbb{R}^n_+$.

then $G(x,y) = \Gamma_x(y) - \Gamma_{x^*}(y) = \Gamma(x-y) - \Gamma(x^*-y)$

Define $\tilde{w}(x) := \int_{B_1^+} G(x,y) f(y)\, dy$. so that $\tilde{w} = 0$ $\{x_n = 0\}$.

(2) Reflect force evenly

$$\tilde{f}(x) = \begin{cases} f(x) & x \in B_1^+ \\ f(x', -x_n) & x \in B_1^- \end{cases}$$



$$\Rightarrow \tilde{w}(x) = \int_{B_1^+} \left(\Gamma(x-y) - \Gamma(x^*-y)\right) f(y)\, dy = \int_{B_1^+} \left(\Gamma(x-y) - \Gamma(x-y^*)\right) f(y)\, dy$$

use radial symmetry

note $\int_{B_1^+} \Gamma(x-y^*) f(y)\, dy = \int_{B_1^-} \Gamma(x-y) \tilde{f}(y)\, dy$ by change of variables.

thus $\tilde{w}(x) = \int_{B_1^+} \Gamma(x-y) f(y)\, dy - \int_{B_1^-} \Gamma(x-y) \tilde{f}(y)\, dy = \boxed{2\int_{B_1^+} \Gamma(x-y) f(y)\, dy - \int_{B_1} \Gamma(x-y) \tilde{f}(y)\, dy}$

the above has two parts! $\tilde{f} \in C_0^{0,\alpha}(B_1)$ so the second part has good estimate

What about first part? let $\overline{w}(x) := \int_{B_1^+} \Gamma(x-y) f(y)\,dy$.

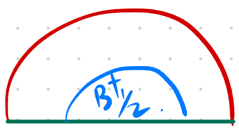
$$\partial_j \overline{w}(x) = \int_{B_1^+} \partial_j \Gamma(x-y)\big(f(y) - f(x)\big)\,dy + f(x) \int_{\partial B_1^+} \partial_i \Gamma(x-y)\, \nu_j \, dS(y)$$

Hölder regularity ensure well-defined.

To see second part,

we use divergence theorem $\int_{B_1^+} \partial_j \Gamma_\varepsilon(x-y)\,dy = \int_{\partial B_1^+} \partial_i \Gamma_\varepsilon(x-y)\, \nu_j \, dS(y)$. ν outward normal then send $\varepsilon \to 0$.

Note $\partial B_1^+ = (\partial B_1 \cap \mathbb{R}^n_+) \cup (B_1 \cap \partial \mathbb{R}^n_+)$



$B_1 \cap \partial \mathbb{R}^n_+$ is what matters.

this doesn't matter since stays $\frac{1}{2}$ distance away.

When $x \in B^+_{1/2}$. (i) But $\nu_j|_{\partial \mathbb{R}^n_+} = 0 \quad \forall j \neq n$. since $\nu = (0, \cdots, 0, -1)$.

$\Rightarrow$ all mixed derivatives contribute 0 on $\partial \mathbb{R}^n_+$.

and the estimate fully inherits from singular integral in B_1^+.

(ii) for pure normal derivative

$$\partial_{nn} \overline{w} = f - \sum_{i=1}^{n-1} \partial_{ii} \overline{w} \qquad \text{use equation!}$$

$$\|\partial_{nn} \overline{w}\|_{C^{0,\alpha}(\overline{B}^+_{1/2})} \lesssim \|f\|_{C^{0,\alpha}} + \sum_{i=1}^{n-1} \|\partial_{ii} \overline{w}\|_{C^{0,\alpha}} \lesssim \|f\|_{C^{0,\alpha}}.$$

To conclude for $\tilde{w}$. $\|\partial_{ij} \tilde{w}\|_{C^{0,\alpha}(\overline{B}^+_{1/2})} \le C(n,\alpha) \|f\|_{C^{0,\alpha}(\overline{B}_1^+)}$

③ for $\vartheta = u - \tilde{w}$. $\begin{cases} \Delta \vartheta = B_1^+ \\ \vartheta = 0 \quad B_1 \cap \{x_n = 0\} \end{cases}$

that $\vartheta = 0$ $\{x_n = 0\}$ is crucial assumption for Schwartz Reflection to preserve harmonicity

Schwartz Reflection (odd reflection) extends ϑ to $\tilde{\vartheta}$ defined in B_1 s.t. $\begin{cases} \Delta \tilde{\vartheta} = 0 & B_1 \\ \tilde{\vartheta} = \vartheta & B_1^+ . \end{cases}$

$$\tilde{\vartheta}(x) := \begin{cases} \vartheta(x) & x \in \overline{B_1^+} \\ -\vartheta(x', -x_n) & x \in B_1^+ \end{cases}$$

easily check MVP is satisfied on $\{x_n = 0\}$!

Now use gradient estimates for harmonic functions.

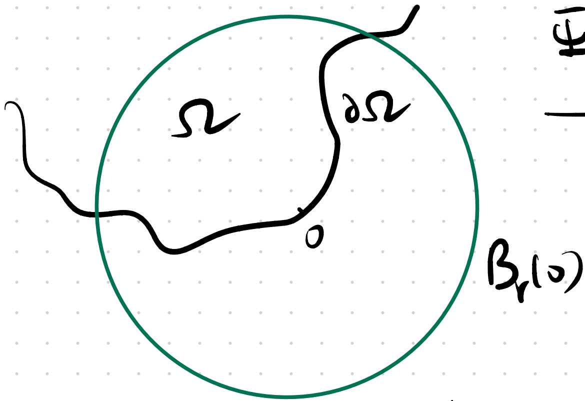
$$\|\vartheta\|_{C^{2,\alpha}(\overline{B}_{1/2}^+)} \le \|\tilde{\vartheta}\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C(n,\alpha) \|\tilde{\vartheta}\|_{C^0(\overline{B}_1)} \le C(n,\alpha) \left(\|u\|_{C^0(\overline{B}_1^+)} + \|f\|_{C^0(\overline{B}_1^+)} \right)$$

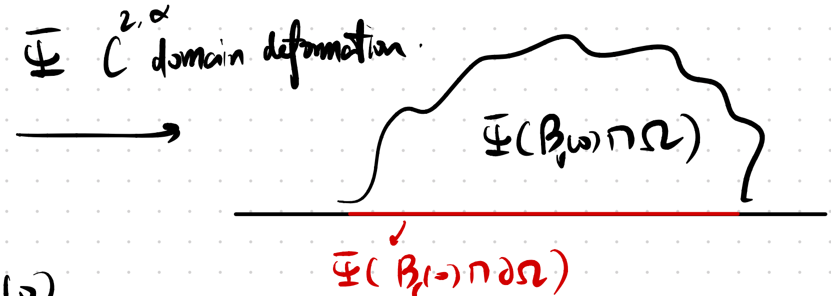
Finally, writing $u = \tilde{w} + \tilde{\vartheta}$, the desired estimate (*) follows ∎

Now we have our Boundary estimates on $\overline{B}_{1/2}^+$.

Can we conclude Global $C^{2,\alpha}$ regularity? **NOT NOW**!

Assume our Ω is $C^{2,\alpha}$.





Sure. this we can always do.

BUT our equation changes !!!

Say $y=\Phi(x)$ Φ $C^{2,\alpha}$ local diffeomorphism.

then assume $\Delta u = f$ in Ω. define $\tilde{u}(y)=u(x)$, $\tilde{f}(y)=f(x)$

So $\partial_i u = \partial_k \tilde{u}\, \partial_i \Phi^k$.

$$\partial_{ij} u = \partial_{kl}\tilde{u}\, \partial_i \Phi^k \partial_j \Phi^l + \partial_k \tilde{u}\, \partial_{ij}\Phi^k$$

and $\tilde{u}$ solves

$$\boxed{\partial_i \Phi^k \partial_i \Phi^l \partial_{kl}\tilde{u} + \Delta \Phi^k \partial_k \tilde{u} = \tilde{f}}$$

this is not Poisson's equation. and currently we can do nothing about it.

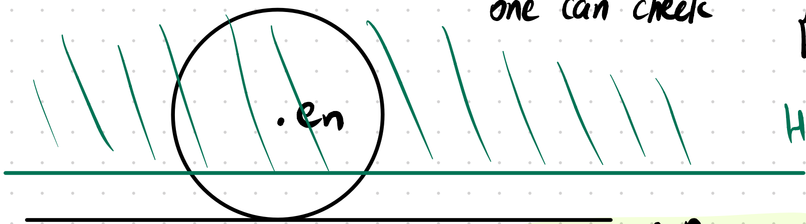
However we can solve on Balls.

Step 2: Global $C^{2,\alpha}$ Regularity on Balls

There is some special Domain transformation Kelvin Transform that preserves the equation!

If $\Omega = B_1(e_n)$ consider $x \in B_1(e_n) \mapsto x^* = \frac{x}{|x|^2}$ reflection point.

one can check $\{x_n > \frac{1}{2}\} = \{x^* \mid x \in B_1(e_n)\}$.



Half plane is "Kelvin Transform" of unit Ball.

moreover consider $u^*(x) := |x|^{2-n} u(x^*)$ $\forall\, x_n > \frac{1}{2}$. (notice $(x^*)^* = x$)

one may check $\Delta u^* = |x|^{-n-2} f(x^*)$. still solves Poisson's equation in $\{x_n > \frac{1}{2}\}$

provided $\Delta u = f$ in $B_1(e_n)$

Thus Lemma 1.

Boundary estimate on Half Ball applies. So Problem (3.1) is solvable

$$\begin{cases} \Delta u = f & B_1 \\ u = g & \partial B_1 \end{cases} \quad f \in C^{0,\alpha}(\bar{B}_1),\ g \in C^{2,\alpha}(\bar{B}_1) \quad \exists!\ u \in C^{2,\alpha}(\bar{B}_1) \quad (**).$$

with $\|u\|_{C^{2,\alpha}(\bar{B}_1)} \le C(n,\alpha)\left(\|\varphi\|_{C^{2,\alpha}} + \|f\|_{C^{0,\alpha}}\right)$

Problem ④

a_{ij} uniformly elliptic, i.e., $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

$$\begin{cases} Lu := a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

we always assume a_{ij} unif. elliptic with elliptic constants $0 < \lambda \le \Lambda < \infty$

Theorem If $a_{ij}, b_i, c, f \in C^{0,\alpha}(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$, Ω $C^{2,\alpha}$ Domain and Maximum Principle holds then $\exists!$ $u \in C^{2,\alpha}(\bar{\Omega})$ solution and $\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C(n, \Omega, \alpha, \lambda, \Lambda)\left(\|f\|_{C^{0,\alpha}(\bar{\Omega})} + \|g\|_{C^{2,\alpha}(\bar{\Omega})}\right)$

Some immediate Remarks

i) $L = \Delta$ with Ω $C^{2,\alpha}$ is a special case, but solves Problem ③ on general $C^{2,\alpha}$ domains.

ii) to prove the theorem, we need the fact that Problem ③ is solvable on Balls.
The ingredient combines $\begin{cases} \text{Method of Continuity} \\ \text{A priori Estimates.} \end{cases}$

iii) that Maximum Principle needs to hold is essential!

- **Method of Continuity**

Let $L_0, L_1 : X \to Y$ be bounded linear operators.

X: Banach space; Y ↳ normed vector space

Consider a family of operators $\mathcal{L}_t := (1-t)L_0 + tL_1 : X \to Y$.

Assume that we know
(i) $\|x\|_X \le C \cdot \|\mathcal{L}_t x\|_Y \quad \forall t \in [0,1], \ \forall x \in X$
(ii) L_0 is invertible, i.e., L_0 bijective, L_0^{-1} bounded linear.

Then L_1 is invertible.

Proof: What do we mean by $\mathcal{L}_t$ invertible? $\forall y \in Y.\ \exists!\ x \in X$ s.t. $\mathcal{L}_t x = y$.

But $\mathcal{L}_t x = y \iff L_0 x = y + t(L_0 - L_1)x$

$\iff x = \underbrace{L_0^{-1} y + t \cdot L_0^{-1}(L_0 - L_1)x}_{\text{Define as } Tx}$ — By assumption (ii) L_0 is invertible

$\Rightarrow$ it suffices to show fixed point exists for T.

Since X is Banach space, we use Contraction Mapping Theorem: it suffices to show $\|T\| < 1$.

$\forall x_1, x_2 \in X$. $\|Tx_1 - Tx_2\|_X \le t \cdot \|L_0^{-1}(L_0 - L_1)(x_1 - x_2)\|_X \overset{\text{(i) estimate on } \mathcal{L}_0^{-1}}{\le} C\, t\, \|(L_0 - L_1)(x_1 - x_2)\|_Y$

$\le C \cdot t \cdot (\|L_0\| + \|L_1\|)\, \|x_1 - x_2\|_X < \|x_1 - x_2\|_X$

By taking $t < \dfrac{1}{2C(\|L_0\| + \|L_1\|)} = \delta$. But this δ is uniform in $x \in X$, $t \in [0,1]$ $\Rightarrow$ push all the way to $t = 1$.

In practice, what are our X, Y, L_0, L_1, and estimate (i) ?

① up to subtracting $u-g$, we treat $\mathcal{L}g \in C^{0,\alpha}(\bar{\Omega})$ as force. Hence WLOG assume $u=0$ $\partial\Omega$

Define $X := C^{2,\alpha}_{zero}(\bar{\Omega}) = \{ u \in C^{2,\alpha}(\bar{\Omega}) \mid u=0 \text{ on } \partial\Omega \}$. $Y := C^{0,\alpha}(\bar{\Omega})$

② $L_0 = \Delta$ Laplace operator is our most natural choice.

$L_1 = \mathcal{L}$ non-divergence form uniformly elliptic operator is our target.

Q What's missing to apply Method of continuity ?

part of our assumption

Reason it's called "A priori"

(i) A priori Estimates. $\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C \|\mathcal{L}_t u\|_{C^{0,\alpha}(\bar{\Omega})}$ $\forall t \in [0,1]$. $\forall u \in C^{2,\alpha}_{zero}(\bar{\Omega})$

Recall $\mathcal{L}_t = (1-t)\mathcal{L}_0 + t\mathcal{L}_1$. So we're really looking for estimates

for u solution to either $\begin{cases} \Delta u = f & \Omega \\ u=0 & \partial\Omega \end{cases}$ $\begin{cases} \mathcal{L}u = f & \Omega \\ u=0 & \partial\Omega \end{cases}$ that $\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C\left(\|u\|_{C^0(\bar{\Omega})} + \|f\|_{C^{0,\alpha}(\bar{\Omega})}\right)$

$\le C \cdot \left(\|f\|_{C^{0,\alpha}(\bar{\Omega})}\right)$ requires maximum principle.

(ii) Invertibility of $L_0 = \Delta$

we've solved Problem ③ on Balls. But we haven't done so for general Domains.

A priori Estimates

part of the assumption!

Theorem Let $u \in C^{2,\alpha}_{zero}(\bar{\Omega})$ be solution to $\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f$ Ω
with a_{ij} unif elliptic, and $a_{ij}, b_i, c, f \in C^{0,\alpha}(\bar{\Omega})$, Ω $C^{2,\alpha}$.
Then $\exists\ C = C(n, \alpha, \Omega, \text{data}) > 0$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C\left(\|u\|_{C^0(\bar{\Omega})} + \|f\|_{C^{0,\alpha}(\bar{\Omega})}\right)$$

notice we do not assume for maximum principle.

Let's only prove for the interior version. Method of Freezing Coefficients.

Lemma Let $u \in C^2(B_1)$ solve $a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f$ B_1. a_{ij} unif elliptic. data $\in C^{0,\alpha}(\bar{B}_1)$
then $\exists\ \delta = \delta(n, \alpha) > 0$ small, and $C = C(n, \alpha, \lambda) > 0$
s.t. whenever $\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}}, \|b_i\|_{C^{0,\alpha}}, \|c\|_{C^{0,\alpha}} \le \delta$ (✱)
one has $\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\bar{B}_1)}\right)$

(✱) is to assume our operator $\mathcal{L}$ is "small perturbation" of Laplace
Hence the estimate should "morally" look the same as that of Laplace's.
Why makes sense to assume? By rescaling! $u_r(x) = u(rx)$ $\forall x \in B_1$.

proof WLOG assume $a_{ij}(0)=\delta_{ij}$. then

$$\boxed{a_{ij}(0)\partial_{ij}u} = \underbrace{(a_{ij}(0)-a_{ij})\partial_{ij}u - b_i\partial_i u - cu + f}_{} =: \boxed{F}$$

this takes the form of Poisson's equation!

So estimate for Laplace writes

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{C^0(\overline{B_{3/4}})} + \|F\|_{C^{0,\alpha}(\overline{B_{3/4}})}\right)$$

In fact, this step uses assumption $u\in C^{2,\alpha}$

$$\le C\Big(\|u\|_{C^0} + \underbrace{\|a_{ij}-\delta_{ij}\|_{C^{0,\alpha}}}_{\le\delta}\|\partial_{ij}u\|_{C^{0,\alpha}} + \underbrace{\|b\|_{C^{0,\alpha}}}_{\le\delta}\|\partial_i u\|_{C^{0,\alpha}} + \underbrace{\|c\|_{C^{0,\alpha}}}_{\le\delta}\|u\|_{C^{0,\alpha}} + \|f\|_{C^{0,\alpha}}\Big)$$

Recall Interpolation for Hölder Spaces. $\forall\varepsilon>0$, $\exists C(\varepsilon)>0$ s.t.

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\varepsilon)\|u\|_{C^0(\overline{\Omega})} + \varepsilon[D^2u]_{C^{0,\alpha}(\overline{\Omega})}$$

So $$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{C^0} + \|f\|_{C^{0,\alpha}}\right) + C\delta[D^2u]_{C^{0,\alpha}(\overline{B_{3/4}})} \qquad (\Delta)$$

it is very annoying that the domain on RHS is larger ······
So we cannot simply absorb.

We can use a trick. $$[D^2u]_{C^{0,\alpha}(\overline{B_{3/4}})} \le C(n)\sup_{B_{1/8}(x)\subseteq B_{3/4}}[D^2u]_{C^{0,\alpha}(\overline{B_{1/8}(x)})}$$

shrink the domain on RHS at the cost of "recentering".

We WLOG assume $\|u\|_{C^0(\overline{B_1})} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \le 1$. By dividing by large constant.

Rescaling $u_r(x) := u(rx)$ $\forall x \in B_1$ we ask how equation u_r solves.

hitting derivatives on $u_r(\frac{x}{r}) = u(x)$ $\forall x \in B_r$ and use $Lu = f$ B_r

to obtain that

$$a_{ij}(rx)\partial_{ij} u_r + r b_i(rx) \partial_i u_r + r^2 c(rx)\, u_r = r^2 f(rx) \quad B_1.$$

my coefficients still satisfy $(\star)$ upon choosing r sufficiently small.

now What happens if I bring my u_r into (Δ) ?

$$r^{2+\alpha}[D^2u]_{C^{0,\alpha}(\overline{B_{r/2}})} \le C\left(\|u\|_{L^\infty(B_{3r/4})} + r^{2+\alpha}\|f\|_{C^{0,\alpha}(\overline{B_{3r/4}})}\right) + \underbrace{C\cdot\delta}_{\mu}\cdot r^{2+\alpha}\cdot \sup_{B_{r/8}(x)\subseteq B_{3r/4}} [D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})}$$

$$\Rightarrow [D^2u]_{C^{0,\alpha}(\overline{B_{r/2}(0)})} \le C\cdot r^{-3} + \mu\cdot[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})} \text{ for some } B_{r/8}(x)\subseteq B_{3r/4}(0)$$

Define sequence $x_0 = 0$.

$$\begin{cases} r_k = \frac{1}{2}\cdot\frac{1}{2^{2k}} & \forall k \ge 0 \\ B_{r_{k+1}}(x_{k+1}) \subseteq B_{3r_k/2}(x_k) & \forall k \ge 1 \end{cases}$$

where $[D^2u]_{C^{0,\alpha}(B_{r_{k+1}}(x_{k+1}))}$ achieves sup.

Define $a_k := [D^2u]_{C^{0,\alpha}(B_{r_k}(x_k))}$. then we obtain

$$\boxed{a_k \le M\cdot 2^{6k} + \mu\cdot a_{k+1} \quad \forall k \ge 0}$$

Recall our goal is to show $\exists\, \mu$ small, $C > 0$ universal

s.t. $a_0 \leq C$ uniformly in $u \in C^{2,\alpha}(B_1)$.

Assume for contradiction $a_0 > C$ for C of our choice.

then $a_k > C\,\sigma^k$ for $\sigma > 1$ universal

- Base Case satisfied
- assume for k.

$$a_{k+1} \geq \frac{1}{\mu}\left(a_k - M \cdot 2^{6k}\right)$$

$$\underset{\text{inductive hypothesis}}{\geq} \frac{1}{\mu}\left(C \cdot \sigma^k - M\, 2^{6k}\right) \underset{\text{want to ensure}}{>} C \cdot \sigma^{k+1}$$

pick for example $\mu = 2^{-7}$.

$$\sigma \in [2^6, 2^7)$$

choose $C > \dfrac{M}{1 - \mu\sigma}$ sufficiently large

so that $\lim_{k \to \infty} a_k = \infty$.

But the limiting point $x_\infty = \lim_{k \to \infty} x_k \in B_{1/2}$ (up to subsequences). Reach Contradiction that $u \in C^{2,\alpha}(B_1)$ ▣

- Boundary a priori estimates follows same procedure from Boundary $C^{2,\alpha}$ estimate for Poisson's Equation on half Balls.
- Global a priori estimates follow by covering with balls and that the class of uniformly elliptic non-divergence form equations with $C^{0,\alpha}$ coefficients remains invariant under $C^{2,\alpha}$ Domain Deformations.

This concludes the proof of Global A priori Schauder estimates

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OK But now. What about Invertibility of $L_0 = \Delta$?

With the help of a priori estimates + method of continuity + Problem (3.1) solvable

we solve Problem (4.1)

If $C^{0,\alpha}$ coefficients, and maximum principle holds

$$\begin{cases} Lu = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f & B_1 \\ u = g & \partial B_1 \end{cases}$$

$$\exists !\ u \in C^{2,\alpha}(\overline{B_1}) \text{ with } \|u\|_{C^{2,\alpha}(\overline{B_1})} \le C(n,\alpha,\lambda,\Lambda)\left(\|f\|_{C^{0,\alpha}(\overline{B_1})} + \|g\|_{C^{2,\alpha}(\overline{B_1})}\right)$$

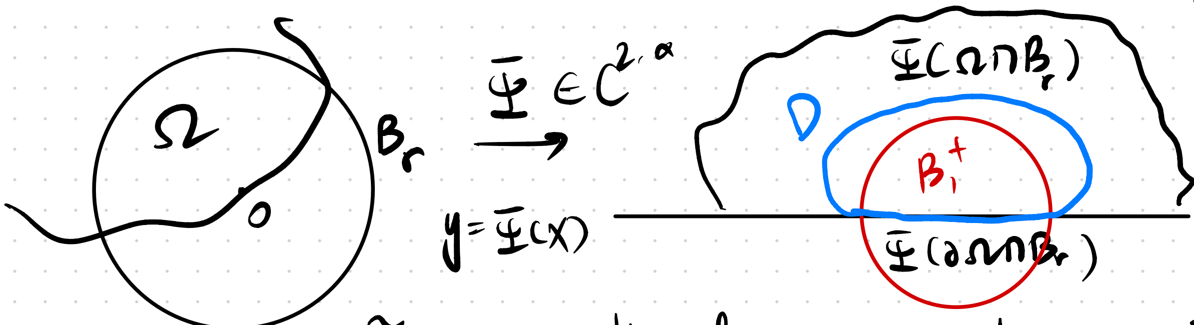
Now. What about general domains Ω that are $C^{2,\alpha}$?

It suffices to ensure $L_0 = \Delta$ is invertible, i.e., for $f \in C^{0,\alpha}(\bar\Omega)$, $g \in C^{2,\alpha}(\bar\Omega)$

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega, \end{cases} \quad \exists!\ u \in C^{2,\alpha}(\bar\Omega) \text{ s.t. } \|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\left(\|f\|_{C^{0,\alpha}(\bar\Omega)} + \|g\|_{C^{2,\alpha}(\bar\Omega)}\right)$$

In other words, it suffices to solve Problem ③ completely.

What do we have? $u \in C^{2,\alpha}(\Omega) \cap C^0(\bar\Omega)$ need to upgrade to Global $C^{2,\alpha}$.



$\tilde u(y) = u(x)$

$\tilde u$ solves $\tilde L \tilde u := \partial_i \Phi^k \partial_i \Phi^l \partial_{kl} \tilde u + \Delta \Phi^k \partial_k \tilde u = \tilde f$ in B_1^+

But what is Boundary Data $\tilde u$ on ∂B_1^+?

We assume only $\tilde u \in C^2(B_1^+) \cap C^0(\bar B_1^+)$

Claim 1 One may solve $\begin{cases} \mathcal{L} u = f & D \\ u = g & \partial D \end{cases}$ for D $C^{2,\alpha}$, D diffeomorphic to a Ball.

Problem (4.2)

this can be done directly with the help of

a priori estimates + Method of Continuity + $L_0 = \Delta$ invertible on D

this is resolved due to Problem (4.1) is solvable

Claim 2 take D as such domain diffeomorphic to Balls that covers B_1^+.

By our assumption $\tilde{u}$ is only continuous up to ∂D. Call $\varphi = \tilde{u}|_{\partial D}$

Now approximate $\{\varphi_k\} \subseteq C^{2,\alpha}(\bar{D})$ boundary data that is uniformly bounded $C^{2,\alpha}(\bar{D})$.

s.t. $\|\varphi_k - \varphi\|_{C^0(\partial D)} \to 0$.

for each φ_k, solve $\tilde{u}_k$

$$\begin{cases} \tilde{\mathcal{L}} \tilde{u}_k = \tilde{f} & D \\ \tilde{u}_k = \varphi_k & \partial D \end{cases} \Rightarrow \exists!\ \tilde{u}_k \in C(\bar{D})$$

this is solvable because Problem (4.2) is solvable.

with $\|\tilde{u}_k\|_{C^{2,\alpha}(\bar{D})} \leq C\left(\|\tilde{f}\|_{C^{0,\alpha}} + \|\varphi_k\|_{C^{2,\alpha}(\bar{D})}\right)$ (✱)

Now, what about this family of $\{\tilde{u}_k\} \subseteq C^{2,\alpha}(\bar{D})$?

- $\begin{cases} \tilde{L}(\tilde{u}_k - \tilde{u}_\ell) = 0 & D \\ \tilde{u}_k - \tilde{u}_\ell = \varphi_k - \varphi_\ell & \partial D \end{cases}$ By Maximum Principle $\|\tilde{u}_k - \tilde{u}_\ell\|_{C^0(\bar{D})} \leq \|\varphi_k - \varphi_\ell\|_{C^0(\partial D)} \to 0$

thus $\exists!\ \tilde{u} \in C^0(\bar{D})$ s.t. $\|\tilde{u}_k - \tilde{u}\|_{C^0(\bar{D})} \to 0$.

which of course coincides with our original $\tilde{u}$.

- But one also has the Estimates (☆)

$$\|\tilde{u}_k\|_{C^{2,\alpha}(\bar{D})} \leq C \quad \text{uniformly bounded in } k.$$

Now by Ascoli-Arzela, the limit $\tilde{u} \in C^{2,\alpha}(\bar{D})$.

In particular $B_1 \cap \{x_n = 0\} \subseteq \partial D$,

So $\|\tilde{u}\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \leq C(n, \alpha, \Phi, \text{data}) \left(\|\tilde{f}\|_{C^{0,\alpha}(\overline{B_1^+})} + \|\tilde{g}\|_{C^{2,\alpha}(\overline{B_1^+})} \right)$

With the Boundary estimates one may go to Global $C^{2,\alpha}$ estimates

$\Rightarrow$ Problem ③ is solvable for $u \in C^{2,\alpha}(\bar{\Omega})$

$\Rightarrow$ Problem ④ is solvable for $u \in C^{2,\alpha}(\bar{\Omega})$

using a priori estimate + method of continuity + Problem ③ solvable

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A Remark on importance of Maximum Principle.

consider for example
$$\begin{cases} u'' + u = 0 & (0, \pi) \\ u(0) = 0 \\ u(\pi) = 1 \end{cases}$$

general solution takes the form

$u(t) = A \sin(t) + B \cos(t)$

But $u(0) = B = 0$

$u(\pi) = -B = 1$

contradiction.

$\Rightarrow$ NO solution to this Dirichlet problem.

Maximum Principle holds in general when

$c \leq 0$ or for some $c_0 = c_0(n, \alpha, \Omega) > 0$, $c \leq c_0$